UNRUFFLED EXTENSIONS AND FLATNESS OVER CENTRAL SUBALGEBRAS

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Abstract. A condition on an affine central subalgebra Z of a noetherian algebra A of finite Gelfand-Kirillov dimension, which we call here *unruffledness*, is shown to be equivalent in some circumstances to the flatness of A as a Z-module. Unruffledness was studied by Borho and Joseph in work on enveloping algebras of complex semisimple Lie algebras, and we discuss applications of our result to enveloping algebras, as well as beginning the study of this condition for more general algebras.

1. Introduction

1.1. Let A be a noetherian algebra, finitely generated over the uncountable algebraically closed field k, and let Z be a finitely generated subalgebra of the centre of A, such that the nonzero elements of Z are not zero divisors in A. The central problem addressed in this paper is: Can we find easily checkable conditions to ensure that A is a flat $Z-$module? Of course this question can and should be approached locally, one maximal ideal of Z at a time, but a global form of our main result (5.1) states:

Theorem. *Let A and Z be as above, and suppose that A is Cohen-Macaulay and that Z is smooth, with $\mathfrak{m}A \neq A$ for all maximal ideals $\mathfrak{m}$ of Z. Then A is a flat $Z-$module if and only if Z is unruffled in A.*

Several terms in the above statement need some explanation, which we give in the next two paragraphs, before turning to motivation and applications.

1.2. Our results and proofs are couched in the setting of algebras of finite Gelfand-Kirillov dimension, denoted $\mathrm{GK-dim}_k(-)$, which we assume exists for all $A-$modules and satisfies various standard desirable properties as listed in (2.1). The *grade* $j_A(M)$ of a finitely generated $A-$module M is defined to be the least integer j such that $\mathrm{Ext}^j_A(M, A)$ is non-zero, or $+\infty$ if no such integer exists; we'll simply write $j(M)$ when the algebra A is clear from the context. The algebra A is *Cohen-Macaulay* if

$$\mathrm{GK-dim}_k(A) \quad = \quad j(M) + \mathrm{GK-dim}_k(M)$$

for all non-zero finitely generated $A-$modules M. (Here and throughout, "module" will mean "left module" when no other qualification is given; so the above definition should strictly speaking be "left Cohen-Macaulay".) To say that Z is *smooth* simply means that Z has finite global (homological) dimension, or equivalently that its maximal ideal space $\mathcal{Z}$ is smooth.

1.3. Let A and Z be as in (1.1), and let $\mathfrak{m}$ be a maximal ideal of Z. Denote the field of fractions of Z by $Q(Z)$. Then Z is said to be *unruffled at* $\mathfrak{m}$ *in* A if

$$\mathrm{GK-dim}_k(A/\mathfrak{m}A) \quad = \quad \mathrm{GK-dim}_{Q(Z)}(A \otimes_Z Q(Z)); \tag{1}$$

and Z is *unruffled in* A (or A is *unruffled over* Z) if (2) holds for all maximal ideals $\mathfrak{m}$ of Z. The concept, although not the name, is due to Borho and Joseph [4, 5.8], who showed there that every prime factor of the enveloping algebra of a complex semisimple Lie algebra is unruffled over its centre. Indeed, following the suggestion of [4, 5.8], a secondary aim of this paper is to begin to investigate the significance of the unruffled hypothesis on an algebra and a central subalgebra. Our reason for proposing the adjective "unruffled" is a result of Borho [6], which shows that the crucial feature of an unruffled extension $Z \subseteq A$ is that

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$\mathrm{GK}-\dim_k(A/\mathfrak{m}A)$ is *constant* as $\mathfrak{m}$ ranges through $\mathcal{Z}$. Because Borho's discussion is set in the specific context of enveloping algebras, we shall derive a version of his result as Lemma (2.3). To do so in the proper generality we need to recall in (2.2) some ideas about generic ideals of algebras over uncountable fields, which go back to work of Borho from the 1970s, [5, Section 4]. Laying out this material in a general setting may have some independent interest.

In (2.4) we discuss a number of examples and non-examples of unruffled extensions $Z \subseteq A$, and explain how our main result collapses to a well-known theorem when A is commutative.

1.4. Pairs $Z \subseteq A$ of algebras satisfying the hypotheses of (1.1) arise naturally and frequently. For example, a flat family of deformations of an affine noetherian algebra B may be exhibited as a pair $Z \subseteq A$ of algebras as in (1.1), with $A/\mathfrak{m}A \cong B$ for some particular maximal ideal $\mathfrak{m}$ of Z, the deformations of B being the algebras $A/\mathfrak{m}'A$ got by varying $\mathfrak{m}$ across $\mathcal{Z}$. Flatness of the family corresponds to A being a flat $Z-$module, so the theorem reveals that, at least in the presence of mild hypotheses on A and Z, this is equivalent to constancy of the GK-dimensions of the deformed algebras.

A second major source of motivating examples is the concept of a stratification of the prime or primitive spectrum of an algebra R into "classically affine strata". The most clearcut examples are given by quantum $n-$space [14], and more generally when $R = \mathcal{O}_q(G)$ is the quantised coordinate ring of a semisimple group G at a generic parameter q, [16, 17]. In these examples the primitive spectrum χ of R is the disjoint union of finitely many locally closed subsets χ_w, and each stratum χ_w is homeomorphic to a torus. The homeomorphism is afforded by induction $\mathfrak{m} \mapsto \mathfrak{m}A_w$, where A_w is a localisation of a factor of R and $\mathfrak{m}$ is a maximal ideal of Z_w, the Laurent polynomial algebra which is the centre of A_w.

In a parallel mechanism, many naturally occurring algebras R which are finite modules over their centres have maximal ideal spectra which can be stratified into finitely many Azumaya strata [10, §5], [11] - including, for example, quantised coordinate rings at a root of unity and symplectic reflection algebras in the PI case. The point we want to make here is not so much that the results of the present paper can contribute anything to an understanding of Azumaya stratifications - they can't! - but rather that some aspects of the Azumaya stratified setting may point towards phenomena which are more generally true. (Recall, for example, that an Azumaya algebra is always projective over its centre.)

A third class of primitive ideal stratifications provides one of our main motivations. Namely, let R be the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional complex semisimple Lie algebra with adjoint group G. In a series of papers Borho, [6, 7, 8, 9], and latterly Borho and Joseph, [4], have studied χ, the space of primitive ideals of R, by defining and studying "generalised Dixmier maps" from subsets of $\mathfrak{g}^*/G$ to subsets of χ. The subsets in question are the *sheets* (of $\mathfrak{g}^*/G$, resp., of χ). The most desirable scenario - sometimes valid, sometimes not - is that a sheet $\mathcal{S}$ in χ should (roughly speaking) consist of the inverse images in R of the ideals of a certain prime factor ring A which are generated by the maximal ideals of the centre Z of A. For a more detailed description of this theory and the relevance of our results to various questions of Borho and Joseph, see (6.3).

1.5. In Section 6 we discuss a number of applications of the main theorem (5.1). In (6.1) we show that a GK-dimension inequality of Smith and Zhang [28], which is used in the proof of (5.1) and which is well-known to be strict in general, is in fact an equality in the presence of the Cohen-Macaulay hypothesis. In (6.2) we derive yet another proof of the theorem of Kostant that the enveloping algebra of a complex semisimple Lie algebra is free over its centre, and develop this into a necessary and sufficient criterion for arbitrary enveloping algebras. As already mentioned, in (6.3) we explore the relevance of (5.1) for Borho and Joseph's work on sheets of primitive ideals. Finally, in (6.4) some preliminary results are proved about the behaviour of the unruffled property under factoring by a centrally generated prime ideal; and on the way some information is produced about the how the Cohen-Macaulay property behaves under factorisation.

1.6. As already indicated, Section 2 contains a discussion of the unruffled property and information about ideals in general position, Section 5 contains the statement and proof of the main theorem, and Section 6 contains applications. The method of proof of the main theorem is homological, and exploits a notion of depth for $Z-A-$bimodules which are finitely generated as $A-$modules. The necessary theory is set up in Section 3, and some technical lemmas on depth in the presence of the unruffled hypothesis are proved in

Section 4. The final short section, Section 7, lists some questions and suggestions for further work arising from the results described in this paper.

2. Unruffled extensions

2.1. **Standing hypotheses.** We'll assume throughout this paper that A denotes an affine noetherian algebra over the algebraically closed field k, and that Z is an affine subalgebra of the centre of A. We assume that Z is a domain whose nonzero elements are not zero divisors in A, as will be the case if, for example, A is prime. We write $Q(Z)$ for the field of fractions of Z. The maximal ideal spectrum of Z will be denoted by $\mathcal{Z}$. The Gelfand-Kirillov dimension over k, denoted $\mathrm{GK}-\dim_k(-)$, will be assumed to exist for all A-modules, and to have the usual desirable properties of being exact and partitive, and taking values in the non-negative integers, as discussed in [20], for example. Let the Gelfand-Kirillov dimensions of A and Z be n and d respectively.

2.2. **Ideals in general position.** As explained in (1.3), in this paragraph and the next we recall some ideas of Borho [5].

In this paragraph we assume that

(2) *k is uncountable and the k-algebra A of (2.1) is finitely related.*

That is, we assume that there exists a free k-algebra $F = k\langle f_1, \dots, f_t\rangle$ of finite rank t and a finitely generated ideal I of F with $F/I \cong A$. Let $r_1, \dots, r_m$ be a set of generators for I, and for $i = 1, \dots, m$ write

$$r_i \quad = \quad \Sigma_{j=1}^{r(i)} \lambda_{ij} \Phi_j$$

where Φ_j are words in the free generators $f_1, \dots, f_t$ of F and $\lambda_{ij} \in k$. Similarly, choose elements $z_1, \dots, z_r$ of F whose images in A generate Z, and write

$$z_s \quad = \quad \Sigma_{u=1}^{e(s)} \mu_{su} \Psi_u$$

for $s = 1, \dots, r$, where Ψ_u are words in $f_1, \dots, f_t$ and $\mu_{su} \in k$. Let k_0 be the prime subfield of k and set $k' = k_0(\lambda_{ij}, \mu_{su} : 1 \leq i \leq m, 1 \leq j \leq r(i), 1 \leq s \leq r, 1 \leq u \leq e(s))$, a countable subfield of k. Set $F' = k'\langle f_1, \dots, f_t\rangle$ and $I' = \sum_{i=1}^n F' r_i F'$, so we can define

$$A' \quad := \quad F'/I';$$

and set Z' to be the k'-subalgebra of A' generated by the images in A' of $z_1, \dots, z_r$. Thus $A = A' \otimes_{k'} k$, so that A' is a prime noetherian affine k'-algebra. Clearly, $Z = Z' \otimes_{k'} k$. (In the case where $Z = Z(A)$ we can simply take $k' = k_0(\lambda_{ij})$ and $Z' = Z(A')$.)

An ideal $\mathfrak{m}$ of Z is said to be *in general position* if $\mathfrak{m} \cap Z' = 0$. Versions of the following results, with similar proofs, were obtained by Borho [5, 4.5c], [6, 2.2, 2.3] for the case when A is a prime factor of a complex semisimple Lie algebra, and with the stronger hypothesis that $Q(Z)A$ is a simple ring for Proposition (2.1).3.[1]

Lemma. *If $\mathfrak{p}$ is a prime ideal of Z in general position then the set*

$$\{\mathfrak{m} \in \mathcal{Z} : \mathfrak{p} \subseteq \mathfrak{m}, \quad \mathfrak{m} \textit{ in general position}\}$$

is dense in $\mathcal{V}(\mathfrak{p}) = \{\mathfrak{m} \in \mathcal{Z} : \mathfrak{p} \subseteq \mathfrak{m}\}$.

Proof. We may assume that $\mathfrak{p}$ is not maximal, so that $\mathcal{V}(\mathfrak{p})$ is an uncountable set. On the other hand, since Z' is countable the set

$$\mathcal{S} \quad := \quad \cup_{z \in Z' \setminus \mathfrak{p}} \mathcal{V}(\mathfrak{p} + zZ)$$

is a countable union of closed proper subsets of $\mathcal{V}(\mathfrak{p})$, and so does not cover $\mathcal{V}(\mathfrak{p})$ by [3, 3.11]. If $\mathcal{V}(\mathfrak{p}) \setminus \mathcal{S}$ were not dense then we'd have covered $\mathcal{V}(\mathfrak{p})$ by a countable union of proper closed subsets, again contradicting [3, 3.11]. So $\mathcal{V}(\mathfrak{p}) \setminus \mathcal{S}$ must be dense in $\mathcal{V}(\mathfrak{p})$. □

[1] In fact there is a problem with part of the argument in [6, 2.2]. Contrary to what is said there, it isn't true that, for an ideal $\mathfrak{p}$ of Z in general position, $Z' \setminus 0$ consists of regular elements *modulo*$(\mathfrak{p}A)$, even when $\mathfrak{p}$ is semiprime, as the example $A = Z = \mathbb{C}[X], \quad A' = Z' = \mathbb{Q}[X], \quad \mathfrak{p} = \langle X(X - \pi)\rangle$ makes plain. Once this is realised, it's not hard to see that [6, Proposition 2.2(1)] is false, and that the best one can say is (using [6, Proposition 2.2(2)]) that if $\mathfrak{p}$ is semiprime with all primes of Z minimal over $\mathfrak{p}$ in general position, then $\mathfrak{p}A$ is semiprime.

Proposition. *Retain the hypotheses and notation introduced in (2.1) and in (2), and let $Q(Z')$ denote the quotient field of Z'.*

1. $A \quad\cong\quad Z \otimes_{Z'} A'$,

and hence

$$Q(Z')A \quad\cong\quad Q(Z')Z \otimes_{Q(Z')} Q(Z')A'. \tag{3}$$

In particular, $Q(Z')A$ is a free module over $Q(Z')Z$, with basis including $\{1\}$, so $Q(Z')Z$ is a direct summand of $Q(Z')A$.

2. Assume that A is semiprime. If $\mathfrak{p}$ is a prime ideal of Z in general position, then $\mathfrak{p}A$ is a semiprime ideal.

3. Assume that A is prime and that $Q(Z)$ is the centre of the simple artinian Goldie quotient ring $Q(A)$. If $\mathfrak{p}$ is a prime ideal of Z in general position, then $\mathfrak{p}A$ is a prime ideal.

4. Assume that $Q(Z)A$ is simple (so A is prime). If $\mathfrak{m}$ is a maximal ideal of Z in general position, then $\mathfrak{m}A$ is a maximal ideal.

Proof. 1. By the associativity of the tensor product,

$$A \quad\cong\quad k \otimes_{k'} A' \quad\cong\quad k \otimes_{k'} Z' \otimes_{Z'} A' \quad\cong\quad Z \otimes_{Z'} A'. \tag{4}$$

Localising these isomorphisms at the central regular elements $Z' \setminus 0$ of A, we find

$$\begin{aligned} Q(Z')A &= Q(Z') \otimes_{Z'} A \\ &\cong Q(Z') \otimes_{Z'} (Z \otimes_{Z'} A') \\ &\cong Q(Z')Z \otimes_{Z'} A' \\ &= (Q(Z')Z \otimes_{Q(Z')} Q(Z')) \otimes_{Z'} A' \\ &\cong Q(Z')Z \otimes_{Q(Z')} Q(Z')A'. \end{aligned}$$

Since $Q(Z')A'$ is a free module over the field $Q(Z')$, the last statement in 1. is immediate from the above isomorphisms.

2. Suppose that A is semiprime, and that $\mathfrak{p}$ is a prime ideal of Z in general position. By (3) and the freeness statement in 1.,

$$Q(Z')A\mathfrak{p} \quad\cong\quad Q(Z')\mathfrak{p} \otimes_{Q(Z')} Q(Z')A', \tag{5}$$

and the elements of $Z' \setminus 0$, being regular *modulo*$(\mathfrak{p})$, are regular *modulo*$(\mathfrak{p}A)$: for notice that, using the last part of 1.,

$$Q(Z')A\mathfrak{p} \cap Z \quad=\quad Q(Z')\mathfrak{p} \cap Z \quad=\quad \mathfrak{p}.$$

Factoring (3) by (5), and abusing notation slightly by writing $Q(Z')(A/\mathfrak{p}A)$ for the partial quotient ring of $A/\mathfrak{p}A$ with respect to the set $(Z' + \mathfrak{p}A/\mathfrak{p}A) \setminus 0_{A/\mathfrak{p}A}$, we obtain the isomorphism in

$$Q(Z')(A/\mathfrak{p}A) \cong Q(Z')(Z/\mathfrak{p}) \otimes_{Q(Z')} Q(Z')A' \subseteq Q(Z/\mathfrak{p}) \otimes_{Q(Z')} Q(Z')A'. \tag{6}$$

The inclusion in (6) again follows by freeness (of $Q(Z')(A/\mathfrak{p}A)$ over $Q(Z')(Z/\mathfrak{p})$), and shows that

$$Q(Z/\mathfrak{p}) \otimes_{Q(Z')} Q(Z')A' \textit{ is a partial quotient ring of } A/\mathfrak{p}A. \tag{7}$$

Since $Q(Z')A'$ is semiprime, so too is $Q(Z/\mathfrak{p}) \otimes_{Q(Z')} Q(Z')A'$, by [12, 3.4.2]. But, from (6), we see that $Q(Z/\mathfrak{p}) \otimes_{Q(Z')} Q(Z')A'$ is generated over $A/\mathfrak{p}A$ by central elements. Hence $A/\mathfrak{p}A$ must also have no non-zero nilpotent ideals, as required.

3. Suppose now that A is prime and that $\mathfrak{p}$ is as in 2. Suppose that $Q(Z)$ is the centre of $Q(A)$. Then $Q(Z')$ is the centre of $Q(A')$, since, by (3),

$$Q(A) \quad\cong\quad Q(Z) \otimes_{Q(Z')} Q(A').$$

So by [25, proof of 7.3.9] $Q(Z/\mathfrak{p}) \otimes_{Q(Z')} Q(A')$ is simple, and hence, being noetherian, it has a simple artinian quotient ring by Goldie's theorem [23]. That $\mathfrak{p}A$ is prime now follows from (7).

4. Suppose now that $\mathfrak{m}$ is a maximal ideal of Z in general position. Since $\mathfrak{m} \cap Z' = 0$, the map $Z \longrightarrow Z/\mathfrak{m}$ induces a homomorphism from $Q(Z')Z$ to k, so $Q(Z') \subseteq k$ and we can form the tensor product $k \otimes_{Q(Z')} Q(Z')A'$. Thus (6) simplifies to

$$A/\mathfrak{m}A \quad=\quad Q(Z')(A/\mathfrak{m}A) \quad\cong\quad k \otimes_{Q(Z')} Q(Z')A'. \tag{8}$$

Suppose now that $Q(Z)A$ is simple. From (4), $Q(Z)A \cong Q(Z) \otimes_{Q(Z')} Q(Z')A'$, so that $Q(Z')A'$ is also simple. Simplicity of $A/\mathfrak{m}A$ follows from this by (8) and [25, proof of 7.3.9]. $\square$

2.3. **Generic constancy of GK-dimension.** As already explained, the following result was obtained by Borho and Joseph for factors of enveloping algebras, with the same proof. It seems reasonable to suspect the truth of a stronger result - namely, that the set of unruffled maximal ideals of Z in A contains a non-empty Zariski-open subset of $\mathcal{Z}$.

Lemma. [4, 5.8] *Keep the hypotheses on Z and A from (2.1) and (2.2), (but there is no need to assume that $Q(Z)A$ is simple). Let $\mathfrak{m}$ be a maximal ideal of Z in general position. Then*

$$\mathrm{GK-dim}_k(A/\mathfrak{m}A) \quad = \quad \mathrm{GK-dim}_{Q(Z)}(Q(Z) \otimes_Z A).$$

Proof. Associativity of the tensor product yields

$$\begin{aligned} Q(Z) \otimes_{Q(Z')} Q(Z')A' &= Q(Z) \otimes_{Q(Z')} Q(Z') \otimes_{Z'} A' \\ &= (Q(Z) \otimes_Z Z) \otimes_{Z'} A' \quad = \quad Q(Z) \otimes_Z A. \end{aligned} \tag{9}$$

From (8) we get

$$\begin{aligned} \mathrm{GK-dim}_k(A/\mathfrak{m}A) &= \mathrm{GK-dim}_{Q(Z')}(Q(Z')A') \\ &= \mathrm{GK-dim}_{Q(Z)}(Q(Z) \otimes_{Q(Z')} Q(Z')A') \\ &= \mathrm{GK-dim}_{Q(Z)}(Q(Z) \otimes_Z A), \end{aligned}$$

where the final equality is given by (9). $\square$

2.4. **Unruffled and ruffled examples.** Recall that, where nothing is said to the contrary, hypotheses (2.1) are assumed to hold throughout.

1. *The case where A is a finitely generated Z-module.* It's clear that if A is a finitely generated Z-module, then A is unruffled over Z. One only needs to note that if $\mathfrak{m}$ is a maximal ideal of Z then $\mathfrak{m}A$ is a proper ideal, which can be seen by inverting the regular elements $Z \setminus \mathfrak{m}$ in A and appealing to Nakayama's lemma.

2. *Prime factors of semisimple enveloping algebras are unruffled over their centres.* If $A = U(\mathfrak{g})/P$ is a prime factor of the enveloping algebra of a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$, then A is an unruffled extension of its centre by [4, Corollary 5.8]. The existing proof of this fact is rather deep, depending as it does on the description of P as induced from a rigid primitive ideal of the enveloping algebra of a Levi subalgebra $\mathfrak{l}$ of a parabolic subalgebra of $\mathfrak{g}$ combined with an irreducible subset of the centre of $\mathfrak{l}$.

3. *The commutative case.* Suppose that all the assumptions of (2.1) hold, but in addition A is commutative and Cohen-Macaulay, so Z is now an arbitrary affine subalgebra of A. Routine local-global yoga applied to [13, Theorem 18.16b and Corollary 13.5] easily yields our main result in this commutative setting: *If $\mathfrak{m}$ is a smooth point of $\mathcal{Z}$, then Z is unruffled in A at $\mathfrak{m}$ if and only if $\mathfrak{m}A \neq A$ and $A_{\mathfrak{m}} := A \otimes_Z Z_{\mathfrak{m}}$ is a flat $Z_{\mathfrak{m}}$-module.*

An instructive example to consider here is the subalgebra $Z = \mathbb{C}[x, xy]$ of the commutative polynomial algebra $A = \mathbb{C}[x, y]$. One easily confirms that, for a maximal ideal $\mathfrak{m}$ of Z, $A_{\mathfrak{m}}$ is a flat $Z_{\mathfrak{m}}$-module if and only if $\mathfrak{m} \neq \langle x, xy \rangle$, while $\mathfrak{m}$ is unruffled in A if and only if $\mathfrak{m} \neq \langle x, xy - \lambda \rangle$, for $\lambda \in \mathbb{C}$.

If one assumes, in addition to the commutativity of A, that A is a finitely generated Z-module, then, noting (2.4).1, one recovers from Theorem (1.1) the familiar fact [13, Corollary 18.17] that a commutative affine Cohen-Macaulay domain is projective over any smooth subring over which it's a finitely generated module.

4. *Enveloping algebras of solvable Lie algebras are not always unruffled over their centres.* Let $\mathfrak{g}$ be the complex solvable Lie algebra with basis x, y, z, t, such that

$$[t, x] = x, \quad [t, y] = -y, \quad [t, z] = -z,$$

and all other brackets are 0. Let $A = U(\mathfrak{g})$ and let Z be the centre of A. Thus $A = R[t; \delta]$ where $R = \mathbb{C}[x, y, z]$ is a commutative polynomial algebra and δ is a derivation. One calculates easily that Z is contained in R, so that Z consists of the δ-invariants in R. Since δ acts semisimply on R, with the eigenvector $x^i y^j z^\ell$ having eigenvalue $i - j - \ell$, it follows that

$$Z = \{\Sigma \mathbb{C} x^i y^j z^\ell : i = j + \ell\} = \mathbb{C}[xz, xy],$$

a polynomial algebra in two variables. For $a, b, \in \mathbb{C}$, let $\mathfrak{m}_{a,b}$ denote the maximal ideal $\langle xy - a, xz - b\rangle$ of Z. It's routine to check that

$$\mathrm{GK}-\dim_{\mathbb{C}}(A/\mathfrak{m}_{a,b}) \quad = \quad 2$$

for $(a, b) \neq (0, 0)$, while $A/\mathfrak{m}_{0,0}$ maps onto $\mathbb{C}[y, z][t; \delta]$, so that

$$\mathrm{GK}-\dim_{\mathbb{C}}(A/\mathfrak{m}_{0,0}) \quad = \quad 3.$$

Thus Z is not unruffled in A at $\mathfrak{m}_{0,0}$.

5. *Left noetherian PI-rings are not always unruffled over their centres.* Let t and s be indeterminates, and define

$$A \quad = \quad \begin{bmatrix} k[t,t^{-1},s] & k[t,t^{-1},s] \\ 0 & k[t] \end{bmatrix},$$

where $k[t, t^{-1}, s]$ is a right $k[t]$-module via the embedding of the second algebra in the first. Thus A is a left noetherian affine PI algebra, but is not semiprime and is not right noetherian. Set Z to be the centre of A, which is easily checked to be the set of scalar matrices and so isomorphic to $k[t]$. Thus $Z \setminus 0$ consists of regular elements of A, and $Q(Z) \cong k(t)$, with

$$A \otimes_Z Q(Z) \quad = \quad \begin{bmatrix} k(t)[s] & k(t)[s] \\ 0 & k(t) \end{bmatrix}.$$

Thus

$$\mathrm{GK}-\dim_{Q(Z)}(A \otimes_Z Q(Z)) \quad = \quad 1.$$

Consider the maximal ideal

$$\mathfrak{m} \quad = \quad \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} Z$$

of Z. One easily calculates that $A/\mathfrak{m}A \cong k$, so that

$$\mathrm{GK}-\dim_k(A/\mathfrak{m}A) = 0.$$

So Z is not unruffled in A at $\mathfrak{m}$.

2.5. **Inequalities for unruffled extensions.** In the presence of flatness the following lemma shows that the strict inequality of GK-dimensions in Example (2.4).5 is the *only* direction in which unruffledness can fail. But Example (2.4).3 (with $\mathfrak{m} = \langle x, xy\rangle$) shows that the flatness hypothesis in the lemma is needed.

Lemma. *Let A and Z be as in (2.1), and let $\mathfrak{m} \in \mathcal{Z}$. Suppose that $A_{\mathfrak{m}}$ is a flat $Z_{\mathfrak{m}}-$module. Then*

$$\mathrm{GK}-\dim_k(A/\mathfrak{m}A) \quad \leq \quad \mathrm{GK}-\dim_{Q(Z)}(A \otimes_Z Q(Z)). \tag{10}$$

Proof. Denote $Q(Z)$ by Q. Let V be a finite dimensional k-vector space which generates A as a k-algebra. So the image $\overline{V}$ of V in $A/\mathfrak{m}A$ [resp. the image of V in $A \otimes_Z Q$] generates $A/\mathfrak{m}A$ [resp. $A \otimes_Z Q$] as a $k-$ [resp. $Q-$] algebra. It will therefore be enough to show that, for all $i \geq 1$,

$$\dim_k(\overline{V}^i) \quad \leq \quad \dim_Q(QV^i). \tag{11}$$

Suppose then that $u_1, \ldots, u_t$ are elements of V^i such that $\Sigma_{j=1}^t q_j u_j = 0$, where $q_j \in Q$, not all zero. We claim that $\overline{u}_1, \ldots, \overline{u}_t$ are k-linearly dependent elements of $\overline{V}^i$. It's clear that this will prove (11).

Multiplying by a suitable element of Z and discarding those u_j for which $q_j = 0$, we get

$$\Sigma_{j=1}^t z_j u_j \quad = \quad 0,$$

with each z_j a non-zero element of Z. Fix a maximal ideal $\mathfrak{m}$ of Z. Choose $\ell \geq 1$, ℓ minimal such that there exists j with $z_j \notin \mathfrak{m}^\ell$. (Note that ℓ exists by the Krull Intersection Theorem [13, Corollary 5.4], Z being a noetherian domain.) Thus

$$\Sigma_{j=1}^t (z_j + \mathfrak{m}^\ell)(u_j + \mathfrak{m}^\ell A) \quad = \quad 0 \tag{12}$$

in $A/\mathfrak{m}^\ell A$, with not all the $z_j + \mathfrak{m}^\ell$ equal to 0. Let $\gamma_1, \ldots, \gamma_p$ be a k-basis for $\mathfrak{m}^{\ell-1}/\mathfrak{m}^\ell$, and write $z_j + \mathfrak{m}^\ell = \Sigma_{r=1}^p \gamma_r \lambda_{jr}$, for $\lambda_{jr} \in k$. Thus (12) gives

$$\Sigma_j(\Sigma_r \gamma_r \lambda_{jr} + \mathfrak{m}^\ell)(u_j + \mathfrak{m}^\ell A) \quad = \quad 0.$$

That is,

$$\Sigma_r \gamma_r (\Sigma_j \lambda_{jr} + \mathfrak{m}^\ell)(u_j + \mathfrak{m}^\ell A) \quad = \quad 0. \tag{13}$$

Now the linear independence of $\{\gamma_r\}$ in $\mathfrak{m}^{\ell-1}/\mathfrak{m}^\ell$ over k implies, thanks to the flatness hypothesis, linear independence of $\{\gamma_r\}$ in $\mathfrak{m}^{\ell-1}A/\mathfrak{m}^\ell A$ over $A/\mathfrak{m}A$. Hence (13) shows that, for each r,

$$\Sigma_{j=1}^t \lambda_{jr} u_j \quad \in \quad \mathfrak{m}A.$$

For some r, there exists j with $\lambda_{jr} \neq 0$. So the result is proved. □

3. Homological equipment

3.1. **Depth.** We need to extend the standard notion of depth from commutative algebra. The classical definition (as in for example [13, page 425]) begins with a commutative noetherian ring R, an ideal I of R, and a finitely generated R-module M with $MI \neq M$, and defines the *depth* of I on M to be the length of a maximal M-sequence of elements of I. (Recall that an *M-sequence* is a sequence $\{x_1, \ldots, x_n\}$ of elements of R such that x_i is not a zero divisor on $M/\Sigma_{j=1}^{i-1} x_j M$, for $i = 1, \ldots, n$; the *length* of the M-sequence is then n.) Crucial to the usefulness of this definition is [13, Theorem 17.4], which guarantees that any two such maximal M-sequences have the same length, and that this number can be read off from an appropriate Koszul complex.

We extend the above definition by allowing the R-module M to be not necessarily finitely generated, but we still insist that $MI \neq M$, and we require M to be an $S-R$-bimodule with S a left noetherian ring and M a finitely generated S-module. With this definition, the analogue of [13, Theorem 17.4], which we state below and prove in (3.4), remains true. Write $R^{(n)}$ for the direct sum of n copies of R. For elements $x_1, \ldots, x_n$ of the commutative noetherian ring R, we denote by $K_R(x_1, \ldots, x_n)$, or by $K(x_1, \ldots, x_n)$ when the ring is clear from the context, the Koszul complex

$$0 \longrightarrow R \longrightarrow R^{(n)} \longrightarrow \wedge^2 R^{(n)} \longrightarrow \cdots \longrightarrow \wedge^i R^{(n)} \xrightarrow{d_{\mathbf{x}}} \wedge^{i+1} R^{(n)} \longrightarrow \cdots \wedge^n R^{(n)} \longrightarrow 0,$$

with $\mathbf{x} = (x_1, \ldots, x_n) \in R^{(n)}$ and $d_{\mathbf{x}}(a) = \mathbf{x} \wedge a$, [13, pages 423-4].

Theorem. *Let R, I, S and M be as stated above, and suppose that $I = \Sigma_{i=1}^n x_i R$. Let r be a non-negative integer. If*

$$H^j(M \otimes_R K(x_1, \ldots, x_n)) \quad = \quad 0$$

for $j < r$, while

$$H^r(M \otimes_R K(x_1, \ldots, x_n)) \quad \neq \quad 0,$$

then every maximal M-sequence in I has length r.

3.2. **Lemma:** *Let R, S and M be as stated in* (3.1), *with $M \neq 0$.*

1. *The set of zero divisors of R on M is equal to the union of a finite set of prime ideals of R.*
2. *If I is an ideal of R which consists of zero divisors on M, then there exists a prime ideal $\mathfrak{p}$ of R with $I \subseteq \mathfrak{p}$, and $0 \neq m \in M$, with $m\mathfrak{p} = 0$.*

Proof. Since M is an $S-R$-bimodule and is left noetherian, it has by [23, Proposition 4.4.9] an affiliated series of prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ as an R-module, in the sense of [23, 4.4.6], such that no element of $R \setminus (\cup_{i=1}^m \mathfrak{p}_i)$ is a zero divisor on M. Thus

$$I \quad \subseteq \quad \cup_{i=1}^m \mathfrak{p}_i.$$

By the prime avoidance property [13, Lemma 3.3] there exists j, $1 \leq j \leq m$, such that $I \subseteq \mathfrak{p}_j$. Since $\{m \in M : m\mathfrak{p}_j = 0\}$ is a non-zero submodule of M, the lemma is proved. □

3.3. For the most part the proof of Theorem (3.1) follows the classical approach as in [13, Section 17.3, proof of 17.4]. Thus [13, Proposition 17.9 and Corollaries 17.10. 17.11] don't involve the module M and so apply unchanged here. But we require an improved version of [13, Corollary 17.12].

Proposition. *Let R, I, S and M be as in (3.1), with $I = \Sigma_{i=1}^{n} x_i R$. Suppose that r is a non-negative integer and that $x_1, \dots, x_r$ is an M-sequence. Then*

$$H^r(M \otimes_R K(x_1, \dots, x_n)) \quad = \quad \{m \in M : mI \subseteq \Sigma_{i=1}^{r} Mx_i\}/\Sigma_{i=1}^{r} Mx_i. \tag{14}$$

Hence, for $j < r$,

$$H^j(M \otimes_R K(x_1, \dots, x_n)) \quad = \quad 0, \tag{15}$$

and if $\{x_1, \dots, x_r\}$ is a maximal M-sequence in I then

$$H^r(M \otimes_R K(x_1, \dots, x_n)) \quad \neq \quad 0. \tag{16}$$

Proof. We prove (14) by induction on r; for $r = 0$, the statement follows from the definition of the Koszul complex. Now suppose that $r > 0$, with the result proved for smaller values of r. We use here induction on n, starting from $n = r$. In this starting case, (14) states that $H^r(M \otimes_R K(x_1, \dots, x_r)) = M/MI$, which is clear from the definition of the Koszul complex.

Suppose now that $n > r$, and the result is known for this r and smaller values of n. By the induction on r we have

$$H^{r-1}(M \otimes_R K(x_1, \dots, x_n)) = \{m \in M : mI \subseteq \Sigma_{i=1}^{r-1} Mx_i\}/\Sigma_{i=1}^{r-1} Mx_i = 0,$$

since x_r is not a zero divisor on $M/\Sigma_{i=1}^{r-1} Mx_i$. Thus the exact sequence of [13, Corollary 17.11] yields

$$\begin{aligned} H^r(M \otimes_R K(x_1, \dots, x_n)) \quad &= \\ &\ker(H^r(M \otimes_R K(x_1, \dots, x_{n-1})) \xrightarrow{x_n \times} H^r(M \otimes_R K(x_1, \dots, x_{n-1})). \end{aligned} \tag{17}$$

Write $N = \{m \in M : m(\Sigma_{i=1}^{n-1} x_i R) \subseteq \Sigma_{i=1}^{r} Mx_i\}$. Then

$$\begin{aligned} \{m \in M : mI \subseteq \Sigma_{i=1}^{r} Mx_i\}/\Sigma_{i=1}^{r} Mx_i \quad &= \\ &\ker((N/\Sigma_{i=1}^{r} Mx_i) \xrightarrow{x_n \times} (N/\Sigma_{i=1}^{r} Mx_i)). \end{aligned} \tag{18}$$

Comparing (17) with (18) proves the induction step for (14).

Since x_{j+1} is not a zero divisor on $M/\Sigma_{i=1}^{j} Mx_i$ for $j < r$, (15) follows at once from (14). To prove (16), suppose that $\{x_1, \dots, x_r\}$ is a maximal M-sequence in I. Then I is contained in the set of zero divisors on $M/\Sigma_{i=1}^{r} Mx_i$. By Lemma (3.2) there exists $m \in M$, $m \notin \Sigma_{i=1}^{r} Mx_i$, such that $mI \subseteq \Sigma_{i=1}^{r} Mx_i$. So (16) follows from this and (14). □

3.4. **Proof of Theorem (3.1).** Let $y_1, \dots, y_s$ be a maximal M-sequence in I. By hypothesis, r is the least integer j such that $H^j(M \otimes_R K(x_1, \dots, x_n)) \neq 0$. Now r is also the least integer j for which $H^j(M \otimes_R K(x_1, \dots, x_n, y_1, \dots, y_s)) \neq 0$, by [13, Corollary 17.10]. Since $MI \neq M$ by hypothesis, Proposition (3.3) shows that $s = r$, proving the theorem. □

3.5. **Definition of depth.** Let R, S, I and M be as in (3.1). Define the *depth* of I on M, denoted $\mathrm{depth}(I, M)$, to be the length of a maximal M-sequence in I. Theorem (3.1) shows that this definition makes sense.

3.6. **Grade versus depth.** We need a noncommutative variant of one of the standard commutative characterisations of depth, as given in [13, Proposition 18.4], for example.

Proposition. *Let S be a noetherian ring with a noetherian central subring R and let J be an ideal of S with $J = (J \cap R)S$. Let N be an $S-S$-bimodule, finitely generated on each side, with R acting centrally on N and $NJ \neq N$. Then the depth of $J \cap R$ on N is equal to the least non-negative integer r such that $\mathrm{Ext}_S^r(S/J, N) \neq 0$.*

Proof. Assume that S, R, N and J are as stated. We prove the theorem by induction on $\operatorname{depth}(J \cap R, N) = t$. Suppose first that $t = 0$. Then it's immediate from Lemma (3.2) that $\operatorname{Hom}_S(S/J, N) \neq 0$, as required.

Now assume that $t \geq 1$ and that the result is proved for smaller values of the depth. Let $x \in J \cap R$ be a regular element on N. We have $J(N/xN) \neq N/xN$, and $\operatorname{depth}(J \cap R, N/xN) = t-1$ by Theorem (3.1). So, by induction, $\operatorname{Ext}_S^{t-1}(S/J, N/xN) \neq 0$, but $\operatorname{Ext}_S^i(S/J, N/xN) = 0$ for all $i < t-1$. Applying $\operatorname{Hom}_S(S/J, -)$ to the exact sequence

$$0 \longrightarrow N \xrightarrow{x\times} N \longrightarrow N/xN \longrightarrow 0,$$

we get for each $j \geq 1$ the exact sequence

$$0 \longrightarrow \operatorname{Ext}_S^{j-1}(S/J, N) \longrightarrow \operatorname{Ext}_S^{j-1}(S/J, N/xN) \longrightarrow \operatorname{Ext}_S^{j}(S/J, N) \longrightarrow 0,$$

where the first and last terms are 0 because $x\operatorname{Ext}_S^i(S/J, N) = 0$ for all i. Hence we deduce that $\operatorname{Ext}_S^i(S/J, N) = 0$ for $i < t$, and $\operatorname{Ext}_S^t(S/J, N) \neq 0$, as required. □

3.7. **Measuring the flat dimension:** The following result is standard and easy for finitely generated modules over a commutative noetherian ring [13, Theorem 6.8], but is false for infinitely generated modules without some additional hypothesis, as can be seen by taking Z to be a polynomial ring in 2 variables and M to be the field of fractions of the factor by a height one prime.

Lemma. *Let A and Z be as in (2.1) and suppose that k is an uncountable field. Let M be a finitely generated A-module which has finite flat dimension t as a Z-module. Then*

$$\begin{aligned} t &= \max\{r : \operatorname{Tor}_Z^r(V, M) \neq 0, V \ a \ Z-module, \dim_k(V) < \infty\} \\ &= \max\{r : \operatorname{Tor}_Z^r(Z/\mathfrak{m}, M) \neq 0, \mathfrak{m} \in \mathcal{Z}\} \\ &= \max\{: \operatorname{Tor}_{Z_\mathfrak{m}}^r(Z/\mathfrak{m}, M_\mathfrak{m}) \neq 0, \mathfrak{m} \in \mathcal{Z}\}. \end{aligned}$$

Proof. The second equality is an easy consequence of the long exact sequence of Tor, and the third is clear since $\mathfrak{m}\operatorname{Tor}_Z^r(Z/\mathfrak{m}, M) = 0$. Since t is finite by hypothesis, it is an upper bound for the right side of the first equality. Moreover the long exact sequence of Tor also shows easily that there exists a prime ideal $\mathfrak{p}$ of Z with $\operatorname{Tor}_Z^t(Z/\mathfrak{p}, M) \neq 0$. Choose $\mathfrak{p}$ to be maximal among such primes, and suppose for a contradiction that $\mathfrak{p}$ is not a maximal ideal. Let $y \in Z \setminus \mathfrak{p}$, with $y + \mathfrak{p}$ not a unit. The exact sequence

$$0 \longrightarrow Z/\mathfrak{p} \xrightarrow{y\times} Z/\mathfrak{p} \longrightarrow Z/\mathfrak{p} + yZ \longrightarrow 0$$

yields

$$\operatorname{Tor}_Z^{t+1}(Z/\mathfrak{p} + yZ, M) \longrightarrow \operatorname{Tor}_Z^t(Z/\mathfrak{p}, M) \xrightarrow{y\times} \operatorname{Tor}_Z^t(Z/\mathfrak{p}, M) \longrightarrow \operatorname{Tor}_Z^t(Z/\mathfrak{p} + yZ, M),$$

in which the two outer terms are zero by our hypotheses on t and $\mathfrak{p}$. Thus multiplication by y is a bijection on $\operatorname{Tor}_Z^t(Z/\mathfrak{p}, M)$; in other words, $\operatorname{Tor}_Z^t(Z/\mathfrak{p}, M)$ is a vector space over the quotient field $Q(Z/\mathfrak{p})$ of $Z/\mathfrak{p}$. But since k is uncountable and $\mathfrak{p}$ is not maximal, $\dim_k(Q(Z/\mathfrak{p}))$ is uncountable. Hence $\dim_k(\operatorname{Tor}_Z^t(Z/\mathfrak{p}, M))$ is also uncountable. This, however, is impossible, since $\operatorname{Tor}_Z^t(Z/\mathfrak{p}, M)$ is a finitely generated module over the countable dimensional k-algebra A. □

4. Unruffled technicalities

4.1. We shall assume throughout Section 4 that A and Z satisfy the hypotheses of (2.1), (so in particular they have GK-dimensions n and d respectively). Recall that the definitions of the Cohen-Macaulay property and of the grade $j(M)$ of an A-module M are given in (1.2).

Lemma. *Let A and Z be as in (2.1) and assume that A is Cohen-Macaulay. Let $\mathfrak{m}$ be a smooth point of $\mathcal{Z}$, and suppose that $\mathfrak{m}A \neq A$ and that*

$$\text{GK}-\dim_k(A/\mathfrak{m}A) \leq \text{GK}-\dim_{Q(Z)}(A \otimes_Z Q(Z)). \tag{19}$$

Then

$$\text{GK}-\dim_k(A/\mathfrak{m}A) = n - d, \tag{20}$$

and Z is unruffled in A at $\mathfrak{m}$.

Proof. The Cohen-Macaulay property of A implies that

$$n \quad = \quad \mathrm{GK}-\dim_k(A/\mathfrak{m}A) + j(A/\mathfrak{m}A), \tag{21}$$

and we note that the validity of (21) is unaffected by inverting the powers of any element of $Z \setminus \mathfrak{m}$ in A, by [20, Proposition 4.2] and the fact that $\mathrm{Ext}^j_A(A/\mathfrak{m}A, A)$ is annihilated by $\mathfrak{m}$ for all j. Similarly, our desired conclusion (20) is clearly unaffected by such a localisation. So we invert in A the powers of an element x of $Z \setminus \mathfrak{m}$, chosen so that in the localised ring $Z[x^{-1}]$, $\mathfrak{m}$ is generated by a regular sequence $\{x_1, \dots, x_d\}$.

By [28, Corollary 2],

$$n \quad = \quad \mathrm{GK}-\dim_k(A) \quad \geq \quad \mathrm{GK}-\dim_{Q(Z)}(A \otimes_Z Q(Z)) + d. \tag{22}$$

By (19) and (22),

$$n - \mathrm{GK}-\dim_k(A/\mathfrak{m}A) \quad \geq \quad d. \tag{23}$$

By (21) and (23),

$$j(A/\mathfrak{m}A) \quad \geq \quad d. \tag{24}$$

In view of Proposition (3.6) we can rewrite (24) as

$$\mathrm{depth}(\mathfrak{m}, A) \quad \geq \quad d. \tag{25}$$

On the other hand the Koszul complex $K_Z(x_1, \dots, x_d)$ gives a Z-free resolution of $Z/\mathfrak{m}$, and applying $- \otimes_Z A$ to this we see that

$$H^d(K_Z(x_1, \dots, x_d) \otimes_Z A) \quad = \quad A/\mathfrak{m}A \quad \neq \quad 0.$$

Thus Theorem (3.1) implies that

$$\mathrm{depth}(\mathfrak{m}, A) \leq d. \tag{26}$$

From (25) and (26), and Proposition (3.6) we find that equality holds in (24); that is, $j(A/\mathfrak{m}A) = d$, and substituting this value in (21) gives (20).

Moreover, substituting (20) in (22) yields

$$\mathrm{GK}-\dim_k(A/\mathfrak{m}A) \geq \mathrm{GK}-\dim_{Q(Z)}(A \otimes_Z Q(Z)), \tag{27}$$

so that, given (19), Z is unruffled in A at $\mathfrak{m}$. □

4.2. **Lemma.** *Let Z and A be as in (2.1), and suppose that Z is unruffled in A. For every prime $\mathfrak{p}$ of Z, $\mathfrak{p}A \cap Z = \mathfrak{p}$.*

Proof. The unruffled hypothesis forces $\mathfrak{m}A \cap Z = \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of Z. If $\mathfrak{p}$ is a prime ideal of Z then

$$\begin{aligned} \mathfrak{p} \quad \subseteq \quad \mathfrak{p}A \cap Z \quad & \subseteq \quad \cap\{\mathfrak{m}A \cap Z : \mathfrak{p} \subseteq \mathfrak{m} \in \mathcal{Z}\} \\ & = \quad \cap\{\mathfrak{m} : \mathfrak{p} \subseteq \mathfrak{m} \in \mathcal{Z}\} \\ & = \quad \mathfrak{p}, \end{aligned}$$

the last equality holding since Z is affine over k [13, Theorem 4.19]. □

4.3. The next result extends one direction of the equality in Lemma (4.1) from maximal to prime ideals of Z. We'll improve both inequalities below to equalities in Theorem (6.4), provided Z is smooth and (2) holds.

Lemma. *Let A and Z be as in (2.1), and suppose that A is Cohen-Macaulay. Let $\mathfrak{p}$ be a prime ideal of Z of height ℓ which is not in the singular locus, and suppose that Z is unruffled in A at the smooth points of $\mathcal{Z}$. Then*

$$\mathrm{GK}-\dim_k(A/\mathfrak{p}A) \geq n - \ell \tag{28}$$

and

$$j(A/\mathfrak{p}A) \leq \ell. \tag{29}$$

Proof. Since A is Cohen-Macaulay of Gelfand-Kirillov dimension n, (28) and (29) are equivalent; we prove (28). Suppose we invert in $A/\mathfrak{p}A$ the powers of an element z of $Z \setminus \mathfrak{p}$; if z is not a zero divisor *modulo*$(\mathfrak{p}A)$ then $\mathrm{GK-dim}_k(A/\mathfrak{p}A)$ is unchanged by this localisation, [20, Proposition 4.2], while if z *is* a zero divisor then $\mathrm{GK-dim}_k(A/\mathfrak{p}A)$ may decrease when we invert z. So in proving (28) we may invert a suitable element of the ideal defining the singular locus and so arrange that Z is smooth. We argue by induction on

$$t \quad := \quad \mathrm{GK-dim}_k(Z/\mathfrak{p}) \quad = \quad d - \ell. \tag{30}$$

The starting point $t = 0$ is given by Lemma (4.1).

Suppose that t is greater than 0, and that we have shown that

$$\mathrm{GK-dim}_k(A/\mathfrak{q}A) \quad \geq \quad n - (\ell + 1) \tag{31}$$

for all primes $\mathfrak{q}$ of height $(\ell + 1)$. We apply Lemma (3.2)(1) with $M = A/\mathfrak{p}A$, which is a non-zero module by Lemma (4.2). The same lemma in fact tells us that $\mathrm{Ann}_Z(M) = \mathfrak{p}$, and since $Z/\mathfrak{p}$ is an affine k-algebra of infinite k-dimension, Lemma (3.2)(1) ensures that there exists $x \in Z$ with $x + \mathfrak{p}$ a non-unit of $Z/\mathfrak{p}$ such that $x + \mathfrak{p}A$ is not a zero divisor in $A/\mathfrak{p}A$. So by [20, Proposition 5.1(e)],

$$\mathrm{GK-dim}_k(A/\mathfrak{p}A + xA) \quad < \quad \mathrm{GK-dim}_k(A/\mathfrak{p}A). \tag{32}$$

But $\mathfrak{p}A + xA = (\mathfrak{p} + xZ)A$, and $\mathrm{GK-dim}_k(Z/\mathfrak{p} + xZ) = t - 1$ by the Principal Ideal Theorem [13, Theorem 10.1 and Corollary 13.4]. Thus the induction hypothesis (31) coupled with (32) yields (28). This proves the induction step and hence the lemma. □

4.4. **Example.** *Lemmas* (4.1) *and* (4.3) *are in general false if* A *is not Cohen-Macaulay.* Consider the Heisenberg group H on 2 generators,

$$H = \langle x, y, : [[x, y], x] = [[x, y], y] = 1 \rangle.$$

Set $z = [x, y]$ and let Z be the subalgebra $k\langle z \rangle$ of the group algebra $A = kH$. Thus Z is the centre of A and clearly A is a free Z-module. By [20, Example 11.10]

$$\mathrm{GK-dim}_k(kH) \quad = \quad 4.$$

One can easily see that, for all maximal ideals $\mathfrak{m}$ of Z,

$$\mathrm{GK-dim}_k(A/\mathfrak{m}A) \quad = \quad 2 \quad = \quad \mathrm{GK-dim}_{Q(Z)}(A \otimes_Z Q(Z)).$$

Thus Lemma (4.1) fails here; clearly A is not Cohen-Macaulay, since, for all maximal ideals $\mathfrak{m}$ of Z, $j(A/\mathfrak{m}A) = 1$.

5. The main theorem

5.1. After stating the result we shall prove the first part in (5.2) and the second in (5.3). Clearly the final part follows from the first two.

Theorem. *Let A and Z satisfy hypotheses (2.1) and suppose that k is an uncountable field. Suppose that A is Cohen-Macaulay. Let I be the defining ideal of the singular locus of Z.*

1. If Z is unruffled in A at the smooth points of $\mathcal{Z}$ then $A[c^{-1}]$ is a flat $Z[c^{-1}]-$module for all non-zero elements c of I.

2. If $\mathfrak{m}$ is a smooth point of $\mathcal{Z}$ such that $\mathfrak{m}A \neq A$ and $A_{\mathfrak{m}}$ is a flat $Z_{\mathfrak{m}}$-module, then Z is unruffled in A at $\mathfrak{m}$.

3. Suppose that Z is smooth and that $\mathfrak{m}A \neq A$ for maximal ideals $\mathfrak{m}$ of Z. Then A is a flat Z-module if and only if Z is unruffled in A.

5.2. **Proof of (5.1)1:** Let $\mathfrak{m}$ be a smooth point of $\mathcal{Z}$. We claim that, for all $i \geq 0$,

$$\mathrm{Tor}^i_Z(Z/\mathfrak{m}, A) \quad = \quad 0. \tag{33}$$

By [28, Corollary 2],

$$\mathrm{GK-dim}_k(A) \quad \geq \quad \mathrm{GK-dim}_{Q(Z)}(Q(Z) \otimes_Z A) + \mathrm{GK-dim}_k(Q(Z)). \tag{34}$$

Now the unruffledness of $\mathfrak{m}$ coupled with (34) yields

$$\mathrm{GK-dim}_k(A) - \mathrm{GK-dim}_k(A/\mathfrak{m}A) \quad \geq \quad d. \tag{35}$$

Since A is Cohen-Macaulay, (35) implies that

$$j(A/\mathfrak{m}A) \quad \geq \quad d. \tag{36}$$

Now Proposition (3.6) shows that there exist elements $x_1, \ldots, x_d$ in $\mathfrak{m}$ forming a regular sequence in A. Set $I = \Sigma_{i=1}^d x_i Z \subseteq \mathfrak{m}$, so that, again by Proposition (3.6),

$$j(A/IA) \quad = \quad d. \tag{37}$$

We claim that

$$\mathfrak{m} \textit{ is minimal over } I. \tag{38}$$

For suppose (38) is false, and let $\mathfrak{p}$ be a prime of Z strictly contained in $\mathfrak{m}$ with $I \subseteq \mathfrak{p}$, so that $\mathfrak{p}$ has height r with $r < d$. Then

$$\mathrm{GK-dim}_k(A/IA) \geq \mathrm{GK-dim}_k(A/\mathfrak{p}A) \geq n - r \geq n - d + 1, \tag{39}$$

where the second inequality is given by Lemma (4.3). But (37) and (39) contradict the fact that A is Cohen-Macaulay, so (38) is true. Localise in Z at $\mathfrak{m}$, so $\mathfrak{m}^t Z_{\mathfrak{m}} \subseteq I_{\mathfrak{m}}$ for some $t \geq 1$. By [13, Corollaries 17.7 and 17.8(a)] $x_1, \ldots, x_d$ constitute a $Z_{\mathfrak{m}}$-sequence in $Z_{\mathfrak{m}}$ since these d elements generate an ideal of the local ring $Z_{\mathfrak{m}}$ containing a $Z_{\mathfrak{m}}$-sequence of length d, namely the tth powers of a regular sequence generating $\mathfrak{m}Z_{\mathfrak{m}}$. Thus the Koszul complex $K_{Z_{\mathfrak{m}}}(x_1, \ldots, x_d)$ gives a free $Z_{\mathfrak{m}}$-resolution of $Z_{\mathfrak{m}}/I_{\mathfrak{m}}$. Since $x_1, \ldots, x_d$ is a regular sequence in A, Proposition (3.3) shows that $K_{Z_{\mathfrak{m}}}(x_1, \ldots, x_d) \otimes_{Z_{\mathfrak{m}}} A_{\mathfrak{m}}$ has no homology except at the dth place. In other words,

$$\mathrm{Tor}^i_{Z_{\mathfrak{m}}}(Z_{\mathfrak{m}}/I_{\mathfrak{m}}, A_{\mathfrak{m}}) \quad = \quad 0$$

for all $i > 0$. Clearly this implies (33). It follows by Lemma (3.7) that, if c is any nonzero element of the ideal defining the singular locus of Z, then $A[c^{-1}]$ is a flat $Z[c^{-1}]$-module, and so (5.1)1 is proved.

5.3. **Proof of (5.1)2:** Suppose that A is Cohen-Macaulay and let $\mathfrak{m}$ be a smooth point of $\mathcal{Z}$ such that $\mathfrak{m}A \neq A$ and $A_{\mathfrak{m}}$ is a flat $Z_{\mathfrak{m}}$-module. Since $Z_{\mathfrak{m}}$ is regular $\mathfrak{m}Z_{\mathfrak{m}}$ is generated by a regular sequence $x_1, \ldots, x_d$. Flatness of the $Z_{\mathfrak{m}}$-module $A_{\mathfrak{m}}$ ensures that $x_1, \ldots, x_d$ is a regular sequence generating $\mathfrak{m}A_{\mathfrak{m}}$, as one can show easily using the Equational Criterion for Flatness [13, Corollary 6.5 and Exercise 6.7]. Therefore

$$j_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}) \quad = \quad d \tag{40}$$

by Proposition (3.6). Since $\mathrm{Ext}^*_A(A/\mathfrak{m}A, A)$ is killed by $\mathfrak{m}$ it follows from (40) that $j_A(A/\mathfrak{m}A) = d$. Hence, since A is Cohen-Macaulay,

$$\mathrm{GK-dim}_k(A/\mathfrak{m}A) \quad = \quad \mathrm{GK-dim}_k(A) - d.$$

Combining this with the inequality (34) of (5.2) yields

$$\mathrm{GK-dim}_k(A/\mathfrak{m}A) \quad \geq \quad \mathrm{GK-dim}_{Q(Z)}(Q(Z) \otimes_Z A).$$

The reverse inequality is supplied by Lemma (2.5), so the proof of (5.1)2 is complete.

5.4. **Examples. 1.** *Theorem (5.1).1 fails to hold whenever A is any affine commutative domain which is not Cohen-Macaulay.* For, given such an algebra A, choose by Noether normalisation [13, Theorem 13.3] a polynomial subalgebra Z over which A is a finitely generated module. So Z is unruffled in A by Example (2.4).**1**. The well-known characterisation of local commutative Cohen-Macaulay algebras by freeness over local smooth subalgebras [13, Corollary 18.17] shows that there must exist a maximal ideal $\mathfrak{m}$ of Z such that $A_{\mathfrak{m}}$ is not a flat $Z_{\mathfrak{m}}$-module.

2. *Theorem (5.1).2 fails in general if A is not Cohen-Macaulay (at least if A is only one-sided noetherian and is not semiprime).* Take A, Z and $\mathfrak{m}$ as in Example (2.4).**5**. Thus A is a flat Z-module, but, as we've already noted, Z is not unruffled in A at $\mathfrak{m}$. Notice that A is not Cohen-Macaulay: $j(A/\mathfrak{m}A) + \mathrm{GK-dim}_k(A/\mathfrak{m}A) = 1 + 0 = 1 < 2 = \mathrm{GK-dim}_k(A)$.

6. Applications

6.1. **The Smith-Zhang inequality.** As noted in [28], the inequality (34), which is their Corollary 2, is in general strict; in fact Example (4.4) is a case where equality fails.[2] However we can deduce easily from Lemma (4.1) that the fact that this example is not Cohen-Macaulay is the key to the failure of the equality in this case:

Corollary. *Let A and Z be as in (2.1), and suppose that A is Cohen-Macaulay. Suppose also that Z has at least one smooth maximal ideal for which $\mathfrak{m}A \neq A$ and*

$$\mathrm{GK}\!-\!\dim_k(A/\mathfrak{m}A) \leq \mathrm{GK}-\dim_{Q(Z)}(A \otimes_Z Q(Z)). \tag{41}$$

For example, if A is finitely related and k is uncountable then this will be the case by Lemma (2.3). Then

$$\mathrm{GK}\!-\!\dim_k(A) \quad = \quad \mathrm{GK}-\dim_{Q(Z)}(Q(Z) \otimes_Z A) + \mathrm{GK}\!-\!\dim_k(Q(Z)). \tag{42}$$

Proof. This is immediate from Lemma (4.1): for this shows that (41) is an equality, both sides being equal to $n-d$. This proves (42). ☐

Remark. The hypothesis that A is Cohen-Macaulay in Corollary (6.1) can be relaxed a little: it is only necessary to assume that there is a Cohen-Macaulay factor A' of A with $\mathrm{GK}\!-\!\dim_k(A') = \mathrm{GK}\!-\!\dim_k(A)$, with the images of the nonzero elements of Z not zero divisors in A'. The adaptations needed to the above argument are obvious.

6.2. **Generalised Kostant theorem:** Let $A = U(\mathfrak{g})$ be the enveloping algebra of a finite dimensional complex Lie algebra $\mathfrak{g}$, and let Z be the centre of A, with $\mathcal{Z} = \mathrm{maxspec}(Z)$ as usual. Example (2.4).4 shows that it's not always true that A is a flat Z-module, even when Z is a polynomial algebra, as one checks in this case by direct calculation or by appealing to Theorem (5.1).2. To state an extra condition needed to ensure flatness, recall that $\mathfrak{g}$ acts on the symmetric algebra $S = S(\mathfrak{g})$ via the adjoint action, and set $Y = S(\mathfrak{g})^{\mathfrak{g}}$. Thus S is the associated graded algebra of the filtered $\mathbb{C}$-algebra A, and the canonical map from A to S is an isomorphism of $\mathfrak{g}$-modules which carries Z to Y and has as inverse the *symmetrisation map*, [12, Proposition 2.4.10].

Theorem. *Retain the above notation, and assume that Z is affine and that $\mathfrak{m}A \neq A$ for all maximal ideals $\mathfrak{m}$ of Z. Let $\mathfrak{y}^+$ be the augmentation ideal of Y, that is $\mathfrak{y}^+ = \mathfrak{g}S \cap Y$.*

1. Consider the statements:

(1) $\mathfrak{y}^+$ is a smooth point of Y and is unruffled in S.
(2) $S_{\mathfrak{y}^+}$ is a flat $Y_{\mathfrak{y}^+}$-module.
(3) Z is unruffled in A at $\mathfrak{m}$ for all smooth points $\mathfrak{m}$ of $\mathcal{Z}$.
(4) $A_{\mathfrak{m}}$ is a flat $Z_{\mathfrak{m}}$-module for all smooth points $\mathfrak{m}$ of $\mathcal{Z}$.

Then

$$(1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4).$$

2. Suppose in addition that Z is smooth (which is equivalent to assuming that Z is a polynomial algebra). If $\mathfrak{y}^+$ is unruffled in S then A is a flat Z-module.

Proof. 1. By [12, Theorem 10.4.5], Y and Z are isomorphic (although not in general via the symmetrisation map); in particular, Y is affine since we are assuming that Z is. Since S is a polynomial algebra and so in particular smooth, $\mathfrak{y}^+$ is a smooth point of Y if $S_{\mathfrak{y}^+}$ is a flat $Y_{\mathfrak{y}^+}$-module, since a finite $S_{\mathfrak{y}^+}$-projective resolution of the trivial S-module yields a finite flat resolution of the unique simple $Y_{\mathfrak{y}^+}$-module. Thus the equivalence of (1) and (2) follows from the commutative case of the Main Theorem (5.1); see (2.4).**3**.

Suppose now that (1) and hence (2) hold. Since the associated graded algebra S of A is smooth, A is Cohen-Macaulay by [1, Theorem II.2.1]. We claim that

$$\mathrm{GK}-\dim_{Q(Z)}(Q(Z) \otimes_Z A) \quad = \quad \mathrm{GK}-\dim_{Q(Y)}(Q(Y) \otimes_Y S). \tag{43}$$

By Corollary (6.1),

$$\mathrm{GK}-\dim_{Q(Z)}(Q(Z) \otimes_Z A) \quad = \quad \mathrm{GK}\!-\!\dim_k(A) - \mathrm{GK}\!-\!\dim_k(Z), \tag{44}$$

[2]For the special case where A is a factor of an enveloping algebra, the inequality was proved in [27].

and similarly (although in this case [13, Theorem 13.5] suffices),

$$\text{GK}-\dim_{Q(Y)}(Q(Y)\otimes_Y S) \quad = \quad \text{GK}-\dim_k(S) - \text{GK}-\dim_k(Y). \tag{45}$$

But S and Y are respectively the associated graded algebras of A and Z, so the right hand sides of (44) and (45) are equal by [20, Proposition 6.6], proving our claim.

Next, as $S_{\mathfrak{y}^+}$ is a flat $Y_{\mathfrak{y}^+}$-module, Lemma (2.5) implies that

$$\text{GK}-\dim_{Q(Y)}(Q(Y)\otimes_Y S) \geq \text{GK}-\dim_k(S/\mathfrak{y}^+ S). \tag{46}$$

Now let $\mathfrak{m}^+ = \mathfrak{g}A\cap Z$, the augmentation ideal of Z. Let $\mathfrak{m}$ be a smooth point of $\mathcal{Z}$. Thus, writing $\text{gr}(-)$ for associated graded modules,

$$\text{gr}(\mathfrak{m}A) \quad \supseteq \quad \text{gr}(\mathfrak{m})S \quad = \quad \text{gr}(\mathfrak{m}^+)S \quad = \quad \mathfrak{y}^+ S. \tag{47}$$

Hence,

$$\text{GK}-\dim_k(S/\mathfrak{y}^+ S) \geq \text{GK}-\dim_k(A/\mathfrak{m}A). \tag{48}$$

By (43), (46) and (48),

$$\text{GK}-\dim_{Q(Z)}(A\otimes_Z Q(Z)) \geq \text{GK}-\dim_k(A/\mathfrak{m}A). \tag{49}$$

But $\mathfrak{m}A \neq A$, so Lemma (2.5) applies and Z is unruffled in A at $\mathfrak{m}$. That is, (2) $\Rightarrow$ (3). The equivalence of (3) and (4) follows from the Main Theorem (5.1).

2. Suppose now that Z is smooth. Thus so also is Y by [12, Theorem 10.4.5]. So the result follows from (1) $\Rightarrow$ (4) of the first part. ☐

Corollary. *(*Kostant, [12, Theorem 8.2.4]*) Suppose that $\mathfrak{g}$ is a finite dimensional complex semisimple Lie algebra. Let $A = U(\mathfrak{g})$ and let Z be the centre of A. Then A is a free Z-module.*

Proof. Retain the notation of the theorem. We check that the hypotheses of the second part of the theorem are satisfied. When $\mathfrak{g}$ is semisimple Z is a polynomial algebra on $\text{rank}(\mathfrak{g})$ indeterminates, [12, Theorem 7.3.8(ii)]. Local finiteness of the adjoint action of $\mathfrak{g}$ on A combined with the semisimplicity of finite dimensional $\mathfrak{g}$-modules imply that A is a direct sum of finitely generated Z-modules, and so $\mathfrak{m}A \neq A$ for each maximal ideal $\mathfrak{m}$ of Z, by Nakayama's Lemma. The subvariety in $\mathfrak{g}^* = \mathfrak{g}$ defined by $\mathfrak{y}^+S(\mathfrak{g})$ is the cone of nilpotent elements, [12, Theorem 8.1.3(i)], which has dimension $\dim_{\mathbb{C}}(\mathfrak{g})-\text{rank}(\mathfrak{g})$ by [12, Theorem 8.1.3(ii)]. So $\mathfrak{y}^+S(\mathfrak{g})$ is unruffled in $S(\mathfrak{g})$. Thus A is a flat Z-module by the second part of the theorem.

Since A is a direct sum of finitely generated (and so projective) $Z-$modules, and projective modules over the polynomial algebra Z are (stably) free, flatness implies freeness in this case. ☐

6.3. **Questions of Borho and Joseph.** Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. In a series of papers [4, 6, 7, 8] Borho and Joseph have studied the primitive spectrum χ of $U(\mathfrak{g})$ by partitioning χ into sheets. By definition, a *sheet* in χ is an irreducible subset $\mathcal{Y}$ of χ which is maximal such that $\text{GK}-\dim_k(U(\mathfrak{g})/P)$ is constant for $P\in\mathcal{Y}$ and the Goldie dimension of $U(\mathfrak{g})/P$ is bounded for $P\in\mathcal{Y}$. In [4, Corollary 5.6] it is shown that every sheet in χ has the form $\overline{\chi}(J,\mathfrak{z})$, where the latter is defined as follows.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ with $\mathfrak{h}\subseteq\mathfrak{p}$ and with Levi decomposition $\mathfrak{p} = \mathfrak{m}\oplus\mathfrak{l}$, and let $\mathfrak{z}$ be the centre of $\mathfrak{l}$. Let J be a primitive completely rigid ideal of $U(\mathfrak{l})$; this means that J is not almost induced[3] from any proper Levi subalgebra of $\mathfrak{l}$. (See [4, 5.6] for details; for example, a primitive ideal of finite codimension is completely rigid, but not conversely in general.) For $\lambda\in\mathfrak{z}^*$, define $I_{\mathfrak{p}}(J,\lambda)$ to be the annihilator in $U(\mathfrak{g})$ of $U(\mathfrak{g})\otimes_{U(\mathfrak{p})}((U(\mathfrak{l})/J)\otimes\mathbb{C}_\lambda)$, where $\mathbb{C}_\lambda$ denotes the one-dimensional $U(\mathfrak{p})-$module with weight λ, where we identify $\mathfrak{z}$ with $\mathfrak{p}/[\mathfrak{p},\mathfrak{p}]$ in order to view $U(\mathfrak{z})-$modules as $U(\mathfrak{p})-$modules. Then

$$\overline{\chi}(J,\mathfrak{z}) \quad = \quad \{I\in\chi : I \textit{ is minimal over } I_{\mathfrak{p}}(J,\lambda), \quad \lambda\in\mathfrak{z}^*\}. \tag{50}$$

Another way of describing $\overline{\chi}(J,\mathfrak{z})$ is as the set of minimal primitive ideals of the prime factor $A = U(\mathfrak{g})/P$ of $U(\mathfrak{g})$, where

$$P \quad = \quad \cap_{\lambda\in\mathfrak{z}^*} I_{\mathfrak{p}}(J,\lambda).$$

[3] A primitive ideal of $U(\mathfrak{l})$) is almost induced if it is a minimal prime over an ideal of the form $I_{\mathfrak{p}'}(J',\mu)$ for a parabolic subalgebra $\mathfrak{p}'$ of $\mathfrak{l}$.

Fix a weight ν such that J is the annihilator of the irreducible highest weight $U(\mathfrak{l})-$module $L'(\nu)$. Here, we can take $\nu \in \mathfrak{z}^{\perp}$, the Killing orthogonal to $\mathfrak{z}$ in $\mathfrak{h}$, so that $\mathfrak{z}^{\perp}$ is a Cartan subalgebra of $[\mathfrak{l}, \mathfrak{l}]$. Then P is the annihilator in $U(\mathfrak{g})$ of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (L'(\nu) \otimes U(\mathfrak{z}))$.

Thus, to study the sheets in χ amounts to studying the collection of minimal primitive ideals of the factors of $U(\mathfrak{g})$ of the form A. In particular, with the notation we've introduced above, the sheet $\overline{\chi}(J, \mathfrak{z})$ consists precisely of (the inverse images in $U(\mathfrak{g})$ of) the prime ideals of A which are minimal over an ideal generated by a maximal ideal of Z, the centre of A. As is implied by Proposition (2.2).4, for a dense set of those maximal ideals $\mathfrak{m}$ of Z, $\mathfrak{m}A$ is in fact prime and hence primitive. However typically there are exceptional $\mathfrak{m}$ for which this is not the case, and in an attempt to remedy this one passes to the larger algebra

$$\tilde{A} \quad := \quad A \otimes_Z \tilde{Z},$$

where $\tilde{Z}$ is the integral closure of Z in its quotient field. It is still not always true that $I = (I \cap \tilde{Z})\tilde{A}$ for every minimal primitive ideal I of $\tilde{A}$ [6, 4.6], but Borho proves in [7, §9, Theorem] that, at least when J is the augmentation ideal of $U(\mathfrak{l})$, every minimal primitive I of $\tilde{A}$ satisfies

$$I \quad = \quad \sqrt{(I \cap \tilde{Z})\tilde{A}}.$$

The analysis of $\tilde{A}$ and $\overline{\chi}(J, \mathfrak{z})$ would be greatly facilitated if a positive answer to the following question from [18] , were known. (See also the closely related question in [4, 5.3, Remark (b)].)

Question 1: Is $\tilde{A}$ a free $\tilde{Z}-$module?

As we've noted in Example (2.4)**2**, A is unruffled, and the same argument from [4, 5.8] shows that

(51) $$\tilde{A} \textit{ is unruffled over } \tilde{Z}.$$

Thus it's clear from Theorem (5.1) that Question 1 is closely connected to

Question 2: With the above notation, is $\tilde{A}$ Cohen-Macaulay?

We don't know the answer to this question. We shall show here however that, at least in an important special case, a positive answer to Question 2 implies a positive answer to Question 1. Retain all the notation already introduced in this subsection. Let $\hat{W}$ be the normaliser of $\mathfrak{z}$ in the Weyl group W of $\mathfrak{g}$, and let $\hat{W}_{\nu} = \{w \in \hat{W} : w\nu = \nu\}$. By [9, Proposition 6.1b] or [4, proof of Proposition 8.6(b)],

$$\tilde{Z} \quad = \quad S(\mathfrak{z})^{(\hat{W}_{\nu}, *)},$$

where $*$ denotes the shifted action, $w * \lambda = w(\lambda + \rho') - \rho'$, and where $\rho' = -\frac{1}{2}(\textit{sum of roots in } \mathfrak{g}/\mathfrak{p}) \mid_{\mathfrak{z}}$. Now assume that

(52) $$\hat{W}_{\nu} \textit{ is generated by reflections,}$$

so that, by the Shepherd-Todd-Chevalley theorem,

(53) $$\tilde{Z} \textit{ is a polynomial algebra.}$$

Theorem. *Retain the notation introduced in this subsection. Assume (52). If $\tilde{A}$ is Cohen-Macaulay, then $\tilde{A}$ is a free $\tilde{Z}-$module.*

Proof. Assume (52) and that $\tilde{A}$ is Cohen-Macaulay. In view of (53) and (51), the hypotheses of Theorem (5.1).3 are satisfied, so we can conclude that $\tilde{A}$ is a flat $\tilde{Z}-$module. Thanks to the local finiteness and complete reducibility of the adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$, $\tilde{A}$ is a direct sum of finitely generated $\tilde{Z}-$modules, so that $\tilde{A}$ is a free $\tilde{Z}-$module as claimed. □

Remarks. 1. When $\mathfrak{g} = \mathfrak{sl}(n)$, $\hat{W}$ is always generated by reflections. Moreover the completely rigid primitive ideal J of $U(\mathfrak{l})$ will always in the $\mathfrak{sl}(n)$ case be co-artinian, [4, 6.10]. Thus if we are concerned only with sheets of completely prime primitive ideals in $U(\mathfrak{sl}(n))$, then J will always be the augmentation ideal of $U(\mathfrak{l})$, so $\nu = 0$, and (52) is satisfied.

2. There are some tentative indications that "many" prime factors of enveloping algebras $U(\mathfrak{g})$ of semisimple Lie algebras may be Auslander-Gorenstein[4] and/or Cohen-Macaulay. For example, if P is a maximal ideal of $U(\mathfrak{g})$ then $U(\mathfrak{g})/P$ is Auslander-Gorenstein by [29]. On the other hand, if P is a minimal primitive ideal then the same conclusion holds by [22]. This latter result can be generalised: if P is any primitive ideal of $U(\mathfrak{g})$ for which (a) $\mathrm{gr}(P)$ is prime and (b) the closure $\overline{\mathcal{O}}$ of the associated (nilpotent) orbit $\mathcal{O}$ of P is Gorenstein, then standard filtered-graded arguments yield that $U(\mathfrak{g})/P$ is Auslander-Gorenstein and Cohen-Macaulay. Sufficient conditions for (a) to hold can be read off from [2]; and (b) always holds for the normalisation of $\overline{\mathcal{O}}$ by [15] or [24], and hence always holds for $\overline{\mathcal{O}}$ itself in type A, [19]. One can then hope to lift such a property from $U(\mathfrak{g})/P$ to the closure of the sheet containing P. But since we have only partial results in this direction we shall not pursue this here.

6.4. **Factors of unruffled algebras.** Suppose that Z and A satisfy (2.1), with Z unruffled in A, and P is (say) a prime ideal of A. Is A/P unruffled over $Z/P \cap Z$? The example below shows that the answer is no even when A is commutative. However it may be that positive results can be obtained in special circumstances; for instance, this might be one route to a more elementary proof of [4, Corollary 5.8], that prime factors of the enveloping algebra of a complex semisimple Lie algebra $\mathfrak{g}$ are unruffled over their centres, since it is relatively easy to see that $U(\mathfrak{g})$ itself is unruffled over its centre, Theorem (6.2).2. In this subsection we show that, at least under some additional hypotheses, the unruffled property is stable under factoring by an ideal of A generated by a prime ideal of Z. On the way we derive some useful subsidiary results.

Example: Let $A = \mathbb{C}[x,y,z]$ and let Z be the subalgebra $\mathbb{C}[x,yz]$ of A. It's trivial to check that Z is unruffled in A; equivalently (by Theorem (5.1).3, A is a flat Z-module. But if we factor by the prime ideal $(x-z)A$ we get the ruffled example (2.4).**3**.

Theorem. *Assume that A and Z satisfy (2.1) and (2) of (2.2). Suppose that Z is smooth, and unruffled in A. Let $\mathfrak{p}$ be a prime ideal of Z of height ℓ. Then:*

(1) *$Z/\mathfrak{p}$ is unruffled in $A/\mathfrak{p}A$.*
Suppose in addition that A is Cohen-Macaulay. Let F denote the field of fractions of $Z/\mathfrak{p}$. Then:
(2) $\mathrm{GK}-\dim_F(A/\mathfrak{p}A \otimes_{Z/\mathfrak{p}} F) \quad = \quad n - d;$
(3) $\mathrm{GK}{-}\dim_k(A/\mathfrak{p}A) \quad = \quad n - \ell.$

Proof. The Main Theorem (5.1).3 implies that A is a flat $Z-$module. Hence, by the Equational Criterion for Flatness, [13, Corollary 6.5 and Exercise 6.7],

$$\text{the elements of } Z \setminus \mathfrak{p} \text{ are not zero divisors in } A/\mathfrak{p}A. \tag{54}$$

In particular, $Z/\mathfrak{p} \subseteq A/\mathfrak{p}A$, and this pair of algebras satisfies the hypotheses (2.1) and (2) of (2.2).

1. Since Z is unruffled in A, the GK-dimension of the factors $(A/\mathfrak{p}A)/(\mathfrak{m}A/\mathfrak{p}A)$ is constant as $\mathfrak{m}$ ranges through the maximal ideals of Z which contain $\mathfrak{p}$. Hence unruffledness of $Z/\mathfrak{p}$ in $A/\mathfrak{p}A$ follows from Lemma (2.3).

2. This is immediate from 1. and Lemma (4.1).

3. The desired result is true when $\mathfrak{p} = 0$ and also, by 2., when $\mathfrak{p}$ is a maximal ideal of Z. Since Z is an affine domain there is a chain $0 = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_\ell = \mathfrak{p} \subset \cdots \subset \mathfrak{p}_d$ of prime ideals of Z. By (54) and [20, Proposition 3.15], the GK-dimension of the factors $A/\mathfrak{p}_iA$ goes down by at least one as we pass up each step of the chain. Since the difference between the GK-dimensions of A and of $A/\mathfrak{p}_dA$ is exactly d, the GK-dimension must go down by exactly one at each step, as required. □

Parts (2) and (3) of the theorem fail without the Cohen-Macaulay hypothesis, as is shown by Example (4.3).

We can improve (3) of the theorem by showing that $A/\mathfrak{p}A$ is GK-homogeneous, but only under the extra - presumably superfluous - hypothesis that A is Auslander-Gorenstein. Recall that a Noetherian ring R is *Auslander-Gorenstein* if the R-module R has finite (equal) right and left injective dimensions, and R satisfies the Auslander conditions; namely, for every non-zero left or right R-module M and every non-negative integer

[4] The definition is recalled in (6.4).

i, every non-zero submodule N of $\mathrm{Ext}^i_R(M,R)$ satisfies $j_R(N) \geq i$. Details and further references can be found in [21], for example.

Lemma. *Let R be a Noetherian Auslander-Gorenstein $k-$algebra, let z be a central element of R, and let V be a non-zero finitely generated $R-$module on which z acts torsion freely. Then $j_R(V) = j_{R[z^{-1}]}(V \otimes_R R[z^{-1}])$.*

Proof. It's clear from the behaviour of Ext$-$groups under central localisation that $j_R(V) \leq j_{R[z^{-1}]}(V \otimes_R R[z^{-1}])$. To prove the reverse inequality, set $\overline{V} = V/Vz$, so that there is an exact sequence

$$0 \longrightarrow V \xrightarrow{z\times} V \longrightarrow \overline{V} \longrightarrow 0.$$

The part of the long exact sequence of Ext-groups around $j := j_R(V)$ is thus

$$\mathrm{Ext}^j_R(\overline{V},R) \longrightarrow \mathrm{Ext}^j_R(V,R) \xrightarrow{z\times} \mathrm{Ext}^j_R(V,R) \longrightarrow \mathrm{Ext}^{j+1}_R(\overline{V},R).$$

Here, $\mathrm{Ext}^j_R(\overline{V},R) = 0$ by [21, Theorem 4.3], since R is Auslander-Gorenstein, showing that $\mathrm{Ext}^j_R(V,R)$ has no $\{z^i\}-$torsion. Thus $j_R(V) \geq j_{R[z^{-1}]}(V \otimes_R R[z^{-1}])$, as required. □

Proposition. *Assume that A and Z satisfy (2.1) and (2) of (2.2). Suppose that A is Auslander-Gorenstein and Cohen-Macaulay, and that Z is smooth, and unruffled in A. Let $\mathfrak{p}$ be a prime ideal of Z of height ℓ. Then there exists an element $z \in Z \setminus \mathfrak{p}$ such that $(A/\mathfrak{p}A)[z^{-1}]$ is Auslander-Gorenstein and Cohen-Macaulay of dimension $n-\ell$..*

Proof. Since Z is smooth, there exists an element z in $Z \setminus \mathfrak{p}$ such that $\mathfrak{p}[z^{-1}]$ is generated by a regular sequence $\{x_1, \ldots, x_\ell\}$ in Z. As in the proof of the lemma, $\{x_1, \ldots, x_\ell\}$ is a regular sequence in A. Thus $(A/\mathfrak{p}A)[z^{-1}]$ is Auslander-Gorenstein by [21, 3.4, Remark (3)].
To prove the Cohen-Macaulay property, let L be a finitely generated $(A/\mathfrak{p}A)[z^{-1}]-$module. By [26, Corollary 11.68]

$$j_{(A/\mathfrak{p}A)[z^{-1}]}(L) \quad = \quad j_{A[z^{-1}]}(L) - \ell. \tag{55}$$

Fix a finitely generated $A-$submodule L_0 of L such that $L = L_0 \otimes A[z^{-1}]$. Then

$$j_{A[z^{-1}]}(L) \quad = \quad j_A(L_0) \tag{56}$$

by the Lemma above and, since A is Cohen-Macaulay,

$$j_A(L_0) \quad = \quad n - \mathrm{GK}-\dim_k(L_0). \tag{57}$$

Since $\mathrm{GK}-\dim_k(L_0) = \mathrm{GK}-\dim_k(L)$ by [20, Proposition 4.2], from (55), (56) and (57) it follows that

$$j_{(A/\mathfrak{p}A)[z^{-1}]}(L) \quad = \quad n - \ell - \mathrm{GK}-\dim_k(L), \tag{58}$$

and this combined with Theorem (6.4) proves the result. □

Corollary. *Assume that A and Z satisfy (2.1) and (2) of (2.2). Suppose that A is Auslander-Gorenstein and Cohen-Macaulay, and that Z is smooth, and unruffled in A. Let $\mathfrak{p}$ be a prime ideal of Z of height ℓ. Then $A/\mathfrak{p}A$ is GK-homogeneous of dimension $n-\ell$, and has an Artinian quotient ring.*

Proof. Let $z \in Z \setminus \mathfrak{p}$ be the element afforded by the proposition. By (54), $A/\mathfrak{p}A$ embeds in $(A/\mathfrak{p}A)[z^{-1}]$, and it is easy to check by a small adjustment to the proof of [20, Proposition 4.2] that it is enough to prove that the desired conclusions hold for $A/\mathfrak{p}A[z^{-1}]$. Now the case of grade zero of the Cohen-Macaulay property implies GK-homogeneity of $(A/\mathfrak{p}A)[z^{-1}]$. That this implies the existence of an Artinian quotient ring for $A/\mathfrak{p}A$ now follows from [21, Theorem 5.3]. □

7. Questions

Some of the questions listed here have already been mentioned earlier; we record them again for the reader's convenience.

7.1. **GK-dimension.** Is there a generalisation of the Main Theorem to settings where GK-dimension is not defined? In particular, is there a good way to define the Cohen-Macaulay property in the absence of GK-dimension?

Example (4.4), the Heisenberg group algebra kH, is Cohen-Macaulay neither with the definition (1.2) used in this paper, nor with the definition using the Krull dimension. Moreover, if one defines *Krull unruffled* extensions in the obvious way, using the Krull dimension rather than the GK-dimension, then kH is *not* Krull unruffled over its centre Z. Nevertheless, kH is free over Z, which of course is smooth and affine. Is there a version of the Main Theorem incorporating this example? For example, perhaps the correct setting is that of algebras A for which there is an integer μ such that $\delta(-) := \mu - j_A(-)$ defines an exact finitely partitive dimension function? By [21, Definition 4.5] this would include Auslander-Gorenstein algebras such as kH.

7.2. **Density of unruffled points.** (Borho-Joseph, [4]; see Lemma (2.3).) Suppose that A and Z satisfy (2.1) and (2) of (2.2). Does the set of unruffled maximal ideals of Z contain a non-empty Zariski open subset of $\mathcal{Z}$?

7.3. **Existence of unruffled extensions.** Find a more elementary proof of the fact that prime factors of the enveloping algebra of a complex semisimple Lie algebra are unruffled over their centres. Find some other large classes of unruffled extensions. For example, what about quantised enveloping algebras $U_q(\mathfrak{g})$ where the quantising parameter q is not a root of unity? Is a prime noetherian affine PI algebra with an affine centre Z always unruffled over Z?

7.4. **Non-central subalgebras.** Let $B \subseteq A$ be any pair of affine noetherian algebras of finite GK-dimension. One can clearly extend the definition of an unruffled extension to this setting: several variants come to mind, but one might try requiring constancy of $\mathrm{GK}-\dim_k(M \otimes_B A)$ as M ranges over all simple right $B-$modules of fixed GK-dimension. Does this lead to an interesting theory? Is there a version of the Main Theorem?

7.5. **Unruffled factors.** Is Corollary (6.4) true without the hypothesis that A is Auslander-Gorenstein? Are all the results of this paragraph valid without assuming Z smooth, provided $\mathfrak{p}$ is a smooth prime?

7.6. **Factors of semisimple enveloping algebras.** First, we repeat (a generalisation of) Joseph's question from [18], already stated in (6.3). Let P be a prime ideal of the enveloping algebra of a complex semisimple Lie algebra $\mathfrak{g}$, and suppose P is induced from a completely rigid primitive ideal J of the enveloping algebra of a parabolic subalgebra $\mathfrak{p}$, $P = I_{\mathfrak{p}}(J, \lambda)$. Let $\tilde{Z}$ be the normalisation of the centre Z of $A = U(\mathfrak{g})/P$. Suppose that $\tilde{Z}$ is smooth. Is $\tilde{A} := A \otimes_Z \tilde{Z}$ a free $\tilde{Z}-$module?

In view of Theorem (6.3) a positive answer to the above question would follow from a positive answer to the following. With the above notation and hypotheses, is $\tilde{A}$ Cohen-Macaulay? One can also ask, of course, whether $\tilde{A}$ is Auslander-Gorenstein.

More generally, the partial results for primitive ideals discussed in (6.3) suggest the following rather wild speculation: if $\tilde{A}$ is the normalisation of an arbitrary prime factor of $U(\mathfrak{g})$, $\mathfrak{g}$ semisimple, is the Auslander-Gorenstein and/or the Cohen-Macaulay property for $\tilde{A}$ controlled by the corresponding property for $\tilde{Z}$? In particular, which primitive factors of $U(\mathfrak{g})$ are Auslander-Gorenstein or Cohen-Macaulay?

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